OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

Specimen Test Two – Issued March 2009

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS.

NOTE: Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.

NAME:
TEST CENTRE:
OXFORD COLLEGE (if known):
DEGREE COURSE:
DATE OF BIRTH:

FOR TEST SUPERVISORS USE ONLY:

[ ] **Tick here if special arrangements were made for the test.**

Please either include details of special provisions made for the test and the reasons for these in the space below or securely attach to the test script a letter with the details.

Signature of Invigilator ______________________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

1. For **ALL APPLICANTS**.

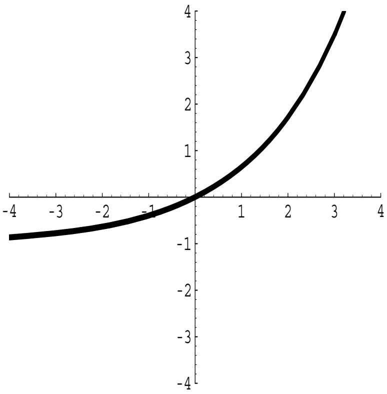
For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. Please show any rough working in the space provided between the parts.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

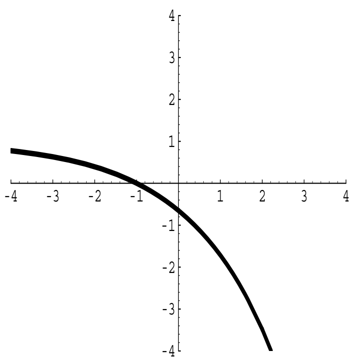
A. The point lying between $P\,(2, 3)$ and $Q\,(8, -3)$ which divides the line PQ in the ratio $1 : 2$ has co-ordinates

(a) $(4, -1)$ (b) $(6, -2)$ (c) $\left(\frac{14}{3}, 2\right)$ (d) $(4, 1)$

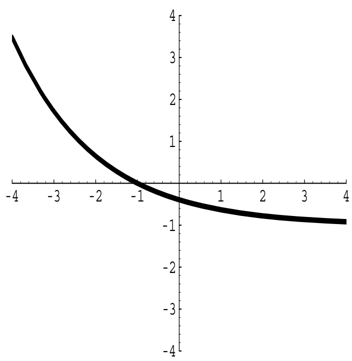
B. The diagram below shows the graph of the function $y = f\,(x)$.



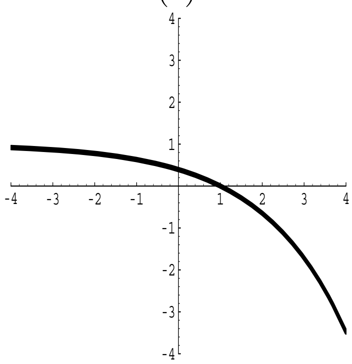
The graph of the function $y = -f\,(x + 1)$ is drawn in which of the following diagrams?



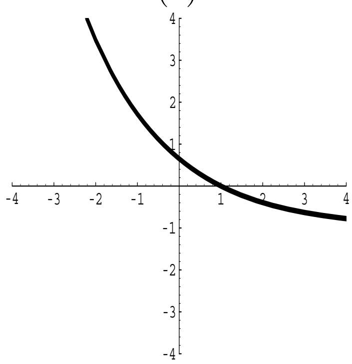
(a)



(b)



(c)



(d)

C. Which of the following numbers is largest in value? (All angles are given in radians.)

(a) $\tan\left(\frac{5\pi}{4}\right)$ (b) $\sin^2\left(\frac{5\pi}{4}\right)$ (c) $\log_{10}\left(\frac{5\pi}{4}\right)$ (d) $\log_2\left(\frac{5\pi}{4}\right)$

D. The numbers x and y satisfy the following inequalities

$$\begin{aligned} 2x + 3y &\leqslant 23, \\ x + 2 &\leqslant 3y, \\ 3y + 1 &\leqslant 4x. \end{aligned}$$

The largest possible value of x is

(a) 6 (b) 7 (c) 8 (d) .9

E. In the range $0 \leqslant x < 2\pi$ the equation

$$\cos(\sin x) = \frac{1}{2}$$

has

(a) no solutions;
(b) one solution;
(c) two solutions;
(d) three solutions.

F. The turning point of the parabola

$$y = x^2 - 2ax + 1$$

is closest to the origin when

(a) $a = 0$ (b) $a = \pm 1$ (c) $a = \pm\frac{1}{\sqrt{2}}$ or $a = 0$ (d) $a = \pm\frac{1}{\sqrt{2}}$.

G. The four digit number 2652 is such that any two consecutive digits from it make a multiple of 13. Another number N has this same property, is 100 digits long, and begins in a 9. What is the last digit of N?

(a) 2 (b) 3 (c) 6 (d) 9

H. The equation

$$\left(x^2+1\right)^{10} = 2x - x^2 - 2$$

(a) has $x = 2$ as a solution;

(b) has no real solutions;

(c) has an odd number of real solutions;

(d) has twenty real solutions.

I. **.**Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$. From these facts, we can deduce that $\log_2 3$, the logarithm of 3 to base 2, is

(a) between $1\frac{1}{3}$ and $1\frac{1}{2}$;
(b) between $1\frac{1}{2}$ and $1\frac{2}{3}$;
(c) between $1\frac{2}{3}$ and 2;
(d) between 2 and 3.

J. Into how many regions is the plane divided when the following three parabolas are drawn?

$$\begin{aligned} y &= x^2 \\ y &= x^2 - 2x \\ y &= x^2 + 2x + 2. \end{aligned}$$

(a) 4 (b) 5 (c) 6 (d) 7

2. For **ALL APPLICANTS.**

Suppose that the equation

$$x^4 + Ax^2 + B = \left(x^2 + ax + b\right)\left(x^2 - ax + b\right)$$

holds for all values of x.

(i) Find A and B in terms of a and b.

(ii) Use this information to find a factorization of the expression

$$x^4 - 20x^2 + 16$$

as a product of two quadratics in x.

(iii) Show that the four solutions of the equation

$$x^4 - 20x^2 + 16 = 0$$

can be written as $\pm\sqrt{7} \pm \sqrt{3}$.

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

Let

$$f(x) = \begin{cases} x+1 & \text{for } 0 \leqslant x \leqslant 1; \\ 2x^2 - 6x + 6 & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

(i) On the axes provided below, sketch a graph of $y = f(x)$ for $0 \leqslant x \leqslant 2$, labelling any turning points and the values attained at $x = 0, 1, 2$.

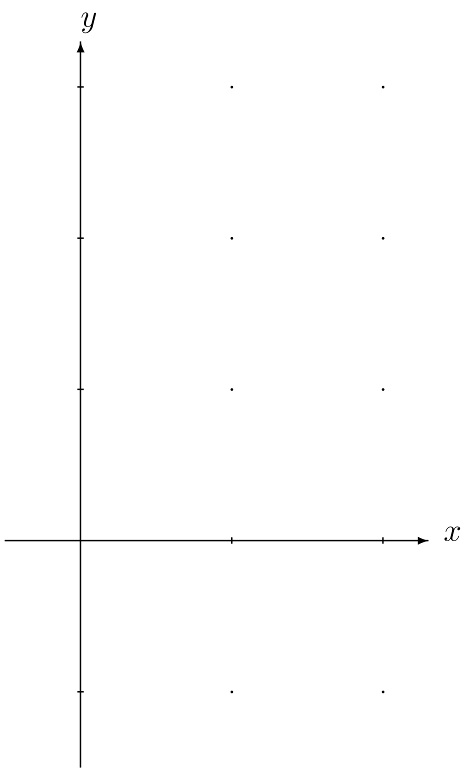
(ii) For $1 \leqslant t \leqslant 2$, define

$$g(t) = \int_{t-1}^{t} f(x)\, \mathrm{d}x.$$

Express $g(t)$ as a cubic in t.

(iii) Calculate and factorize $g'(t)$.

(iv) What are the minimum and maximum values of $g(t)$ for t in the range $1 \leqslant t \leqslant 2$?



4.

For **APPLICANTS IN** { **MATHEMATICS** / **MATHEMATICS & STATISTICS** / **MATHEMATICS & PHILOSOPHY** } **ONLY.**

Mathematics & Computer Science and *Computer Science* **applicants should turn to page 14.**

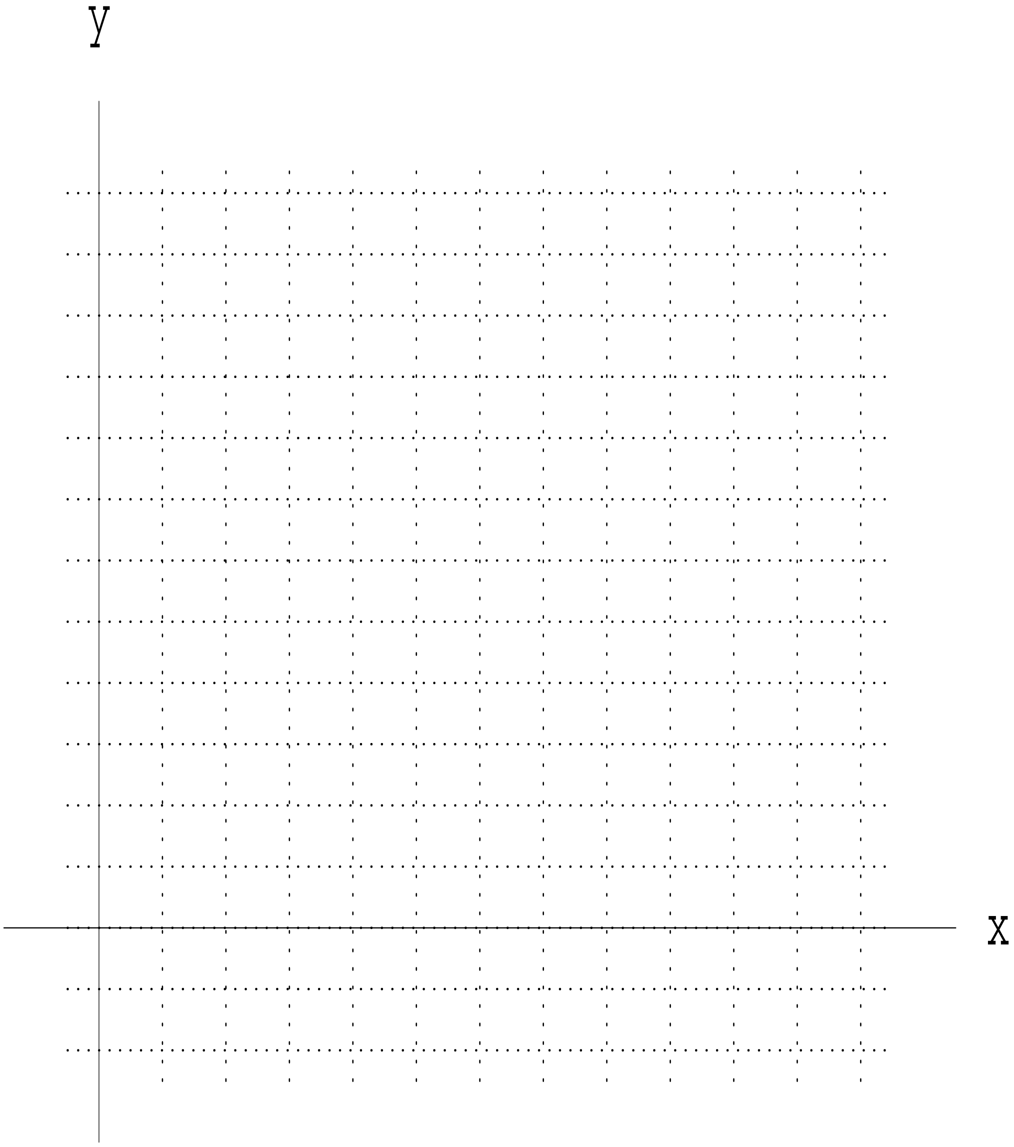
Let P and Q be the points with co-ordinates $(7, 1)$ and $(11, 2)$.

(i) The mirror image of the point P in the x-axis is the point R with co-ordinates $(7, -1)$. Mark the points P, Q and R on the grid provided opposite.

(ii) Consider paths from P to Q each of which consists of two straight line segments PX and XQ where X is a point on the x-axis. Find the length of the shortest such parth, giving clear reasoning for your answer. (You may refer to the diagram to help your explanation, if you wish.)

(iii) Sketch in the line ℓ with equation $y = x$. Find the co-ordinates of S, the mirror image in the line ℓ of the point Q, and mark in the point S.

(iv) Consider paths from P to Q each of which consists of three straight line segments PY, YZ and ZQ, where Y is on the x-axis and Z is on the line ℓ. Find the shortest such path, giving clear reasoning for your answer.


y
x

5. For **ALL APPLICANTS.**

An $n \times n$ square array contains 0s and 1s. Such a square is given below with $n = 3$.

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix}$$

Two types of operation C and R may be performed on such an array.

- The first operation C takes the first and second columns (on the left) and replaces them with a single column by comparing the two elements in each row as follows; if the two elements are the same the C replaces them with a 1, and if they differ C replaces them with a 0.
- The second operation R takes the first and second rows (from the top) and replaces them with a single row by comparing the two elements in each column as follows; if the two elements are the same the R replaces them with a 1, and if they differ R replaces them with a 0.

By way of example, the effects of performing R then C on the square above are given below.

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix} \xrightarrow{R} \begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \xrightarrow{C} \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$$

(a) If R then C are performed on a 2×2 array then only a single number (0 or 1) remains.

(i) Write down in the grids on the next page the eight 2×2 arrays which, when R then C are performed, produce a 1.

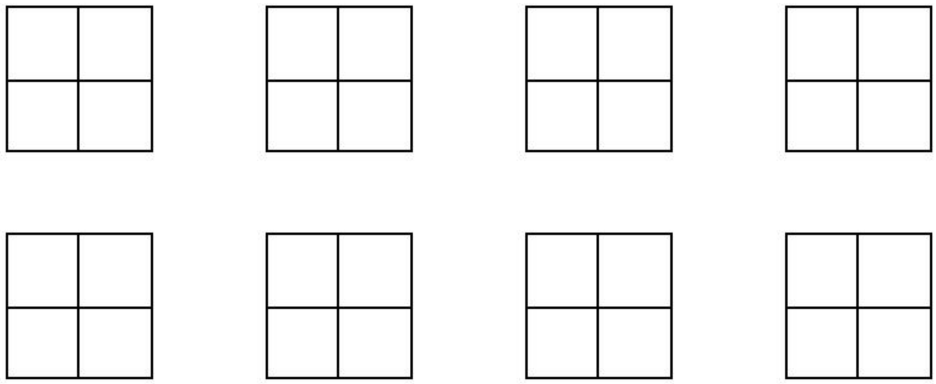
(ii) By grouping your answers accordingly, show that if $\begin{matrix} a & b \\ c & d \end{matrix}$ is amongst your answers to part (i) then so is $\begin{matrix} a & c \\ b & d \end{matrix}$.

Explain why this means that doing R then C on a 2×2 array produces the same answer as doing C first then R.

(b) Consider now a $n \times n$ square array containing 0s and 1s, and the effects of performing R then C or C then R on the square.

(i) Explain why the effect on the right $n - 2$ columns is the same whether the order is R then C or C then R. [This then also applies to the bottom $n - 2$ rows.]

(ii) Deduce that performing R then C on an $n \times n$ square produces the same result as performing C then R.



6.

For **APPLICANTS IN** { **COMPUTER SCIENCE** / **MATHEMATICS & COMPUTER SCIENCE** } **ONLY.**

(i) Alice, Bob, Charlie and Dianne each make the following statements:

Alice:	I am telling the truth.
Bob:	Alice is telling the truth.
Charlie:	Bob is telling the truth.
Dianne:	Charlie is lying.

Only one of the 4 people is telling the truth. Which one? Explain your answer.

(ii)They now make the following statements:

Alice:	Bob is lying.
Bob:	Charlie is lying.
Charlie:	I like beer.
Dianne:	2+2=4.

Now two of the four people are telling the truth. Which two? Explain your answer.

(iii) They are now joined by Egbert. They each make the following statements:

Alice:	I like wine.
Bob:	Charlie is lying.
Charlie:	Alice is lying.
Dianne:	Alice likes beer.
Egbert:	Alice likes beer.

Now three of the five people are telling the truth. Which ones? Explain your answer.

7.

For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

Suppose you have an unlimited supply of black and white pebbles. There are four ways in which you can put two of them in a row: BB, BW, WB and WW.

(i) Write down the eight different ways in which you can put three pebbles in a row.

(ii) In how many different ways can you put N pebbles in a row?

Suppose now that you are not allowed to put black pebbles next to each other. There are now only three ways of putting two pebbles in a row, because BB is forbidden.

(iii) Write down the five different ways that are still allowed for three pebbles.

Now let r_N be the number of possible arrangements for N pebbles in a row, still under the restriction that black pebbles may not be next to each other, so $r_2 = 3$ and $r_3 = 5$.

(iv) Show that for $N \geq 4$ we have $r_N = r_{N-1} + r_{N-2}$. Hint: consider separately the case where the last pebble is white, and the case where it is black.

Finally, suppose that we impose the further restriction that the first pebble and the last pebble cannot both be black. Let w_N be the number of such arrangements for N pebbles; for example, $w_3 = 4$, since the configuration BWB is now forbidden.

(v) For $N \geq 5$, write down a formula for w_N in terms of the numbers r_i, and explain why it is correct.

End of Last Question